

Phase Transition for Strongly Correlated Percolation Models on Supercritical Bernoulli Clusters

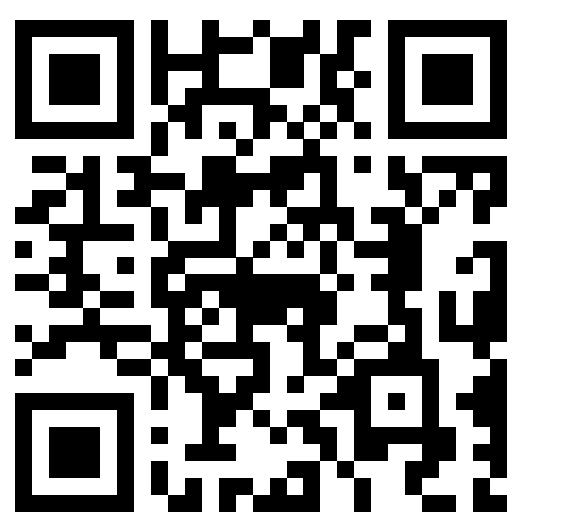
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Chiarini, Liu, and
Nitzschner 2026

Gaussian free field and interlacements with disorder

Let $d \geq 3$. Consider Bernoulli bond percolation with law $\mathbb{Q}_p$ and $p > p_c(d)$. Let $\mathcal{C}_\infty(\omega)$ be the unique infinite open cluster. We fix a typical environment ω and consider two models on $\mathcal{C}_\infty$. It is known that a typical $\mathcal{C}_\infty$ is transient so that both models can be naturally defined.

Gaussian free field on $\mathcal{C}_\infty$: A centered Gaussian field $(\varphi_x)_{x \in \mathcal{C}_\infty}$ with covariance

$$\mathbb{E}_\omega^G[\varphi_x \varphi_y] = g^\omega(x, y), \text{ for } x, y \in \mathcal{C}_\infty,$$

where g^ω is the Green function of the simple random walk on $\mathcal{C}_\infty$. Its level set at height $\alpha \in \mathbb{R}$ is

$$E_\omega^{\geq \alpha} = \{x \in \mathcal{C}_\infty : \varphi_x \geq \alpha\}.$$

Random interlacements on $\mathcal{C}_\infty$: The trace of a Poisson cloud of doubly infinite random walk trajectories η on $\mathcal{C}_\infty$, with intensity parameter $u \geq 0$. It is denoted by $\mathcal{I}_\omega^u = \mathcal{I}_\omega^u(\eta)$ and its vacant set is

$$\mathcal{V}_\omega^u = \mathcal{C}_\infty \setminus \mathcal{I}_\omega^u, \text{ satisfying } \mathbb{P}_\omega^I[K \subseteq \mathcal{V}_\omega^u] = \exp(-u \cdot \text{cap}^\omega(K)), \text{ for } K \subset \subset \mathcal{C}_\infty.$$

Here $\text{cap}^\omega(K)$ denotes the random walk capacity of K on $\mathcal{C}_\infty$.

Percolation question. Do non-trivial percolation phase transitions persist despite the irregular local geometry of the underlying cluster? We know that both models exhibit non-trivial phase transitions on $\mathbb{Z}^d$, $d \geq 3$, see Bricmont, Lebowitz, and Maes 1987; Rodriguez and Sznitman 2013; Sznitman 2010; Sidoravicius and Sznitman 2009.

Theorem 1 (Phase transition for level sets).

There exists a deterministic critical value $\alpha_*(d, p) \in [0, \infty)$ such that $\mathbb{Q}_p$ -almost surely, for each $\alpha < \alpha_*(d, p)$, $E_\omega^{\geq \alpha}$ contains an infinite cluster $\mathbb{P}_\omega^G$ -almost surely, while for each $\alpha > \alpha_*(d, p)$, $E_\omega^{\geq \alpha}$ contains only finite clusters $\mathbb{P}_\omega^G$ -almost surely.

Difficulties & Proof ideas

- **Non-randomness of the critical values.** One can define the quenched critical values $\alpha_*^\omega, u_*^\omega$, show their translation invariance, and conclude by the ergodicity of $\mathbb{Q}_p$.
- **Long-range correlations.** We aim at using a renormalization scheme developed for $\mathbb{Z}^d$ (see Fig. 2, and Sznitman 2010; Rodriguez and Sznitman 2013; Ráth 2015). Note that an infinite path in $\mathcal{C}_\infty$ is also an infinite path in $\mathbb{Z}^d$.

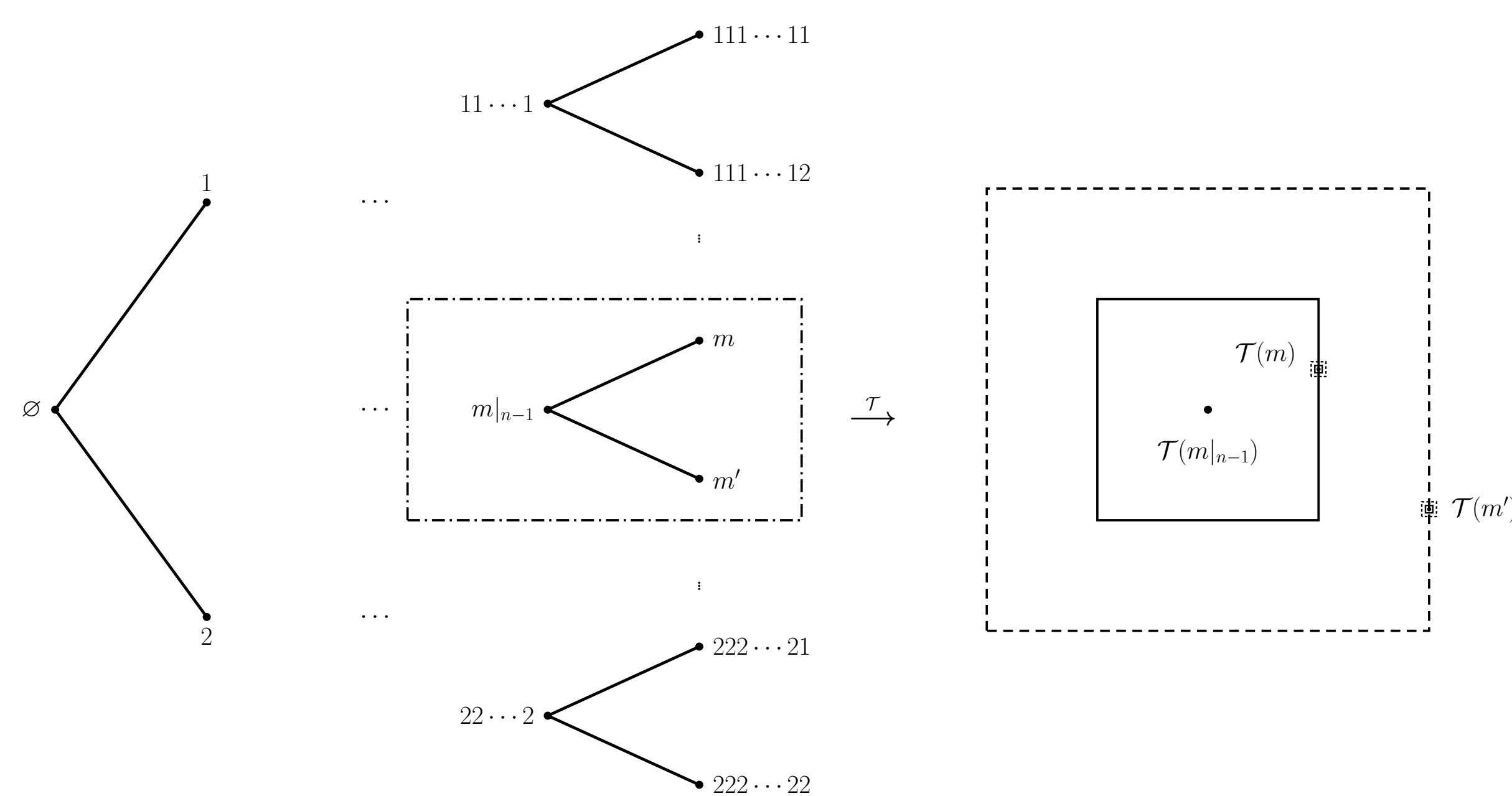


Fig. 2: A crossing at scale $L_n = L_0 \ell_0^n$ (with $L_0 \geq 1, \ell_0 \geq 100$) forces crossings in 2^n boxes at the base scale. The binary tree indexes these boxes (left), and its embedding specifies their spatial locations (right).

We have, in the Gaussian free field case, when proving $\alpha_* < \infty$,

$$\mathbb{P}_\omega^G[S(0, L_n - 1) \xrightarrow{E_\omega^{\geq \alpha}} \infty] \leq \mathcal{C}_{d, \ell_0}^{2^n} \max_{\text{all trees } \mathcal{T}} \mathbb{P}_\omega^G \left[\bigcap_{\text{all leaves } m} z \in B(\mathcal{T}(m), L_0) \cap \mathcal{C}_\infty \right] \bigcup \{z \in E_\omega^{\geq \alpha}\} \Big].$$

In the random interlacements case, one can replace G by I and $E_\omega^{\geq \alpha}$ by $\mathcal{V}_\omega^u$. Here, $\mathcal{C}_{d, \ell_0}^{2^n}$ is an upper bound for the number of admissible tree embeddings.

- **Bad geometry at the base scale I.** This problem appears in the proof of the existence of both regimes. We take the proof of $\alpha_* < \infty$ as an example.

– **sup $_{x \in \mathcal{C}_\infty} g^\omega(x, x) = \infty$.** It is typical for some leaf-boxes to have points above level α , which do not contribute to the probabilistic cost.

– **Uniform estimate over all tree embeddings.** We show that for fixed $\rho \in (0, 1)$, a set of **approximately local** Bernoulli random variables $(\zeta_x)_{x \in L_0 \mathbb{Z}^d}$, $\mathbb{Q}_p$ -a.e. ω ,

$$\sup_{\text{all trees } \mathcal{T}} \frac{1}{2^n} \sum_{\text{all leaves } m} \zeta_{\mathcal{T}(m)}(\omega) < \rho \text{ for all but finitely many } n.$$

– **Encoding local degeneracy.** We apply the above result to the random variables introduced by Barlow 2004 that encode the local degeneracy scale of the percolation cluster. Combined with Barlow and Hambly 2009, we are able to derive upper bounds on the probability that a given set is contained in the level set.

- **Bad geometry at the base scale II.** (Note that for the Gaussian free field, $\alpha_* \geq 0$ follows from a general result for locally finite transient weighted connected graphs, see Abächerli and Sznitman 2018.) This problem appears in the proof of the existence of a supercritical regime for the vacant set of random interlacements ($u_* > 0$).

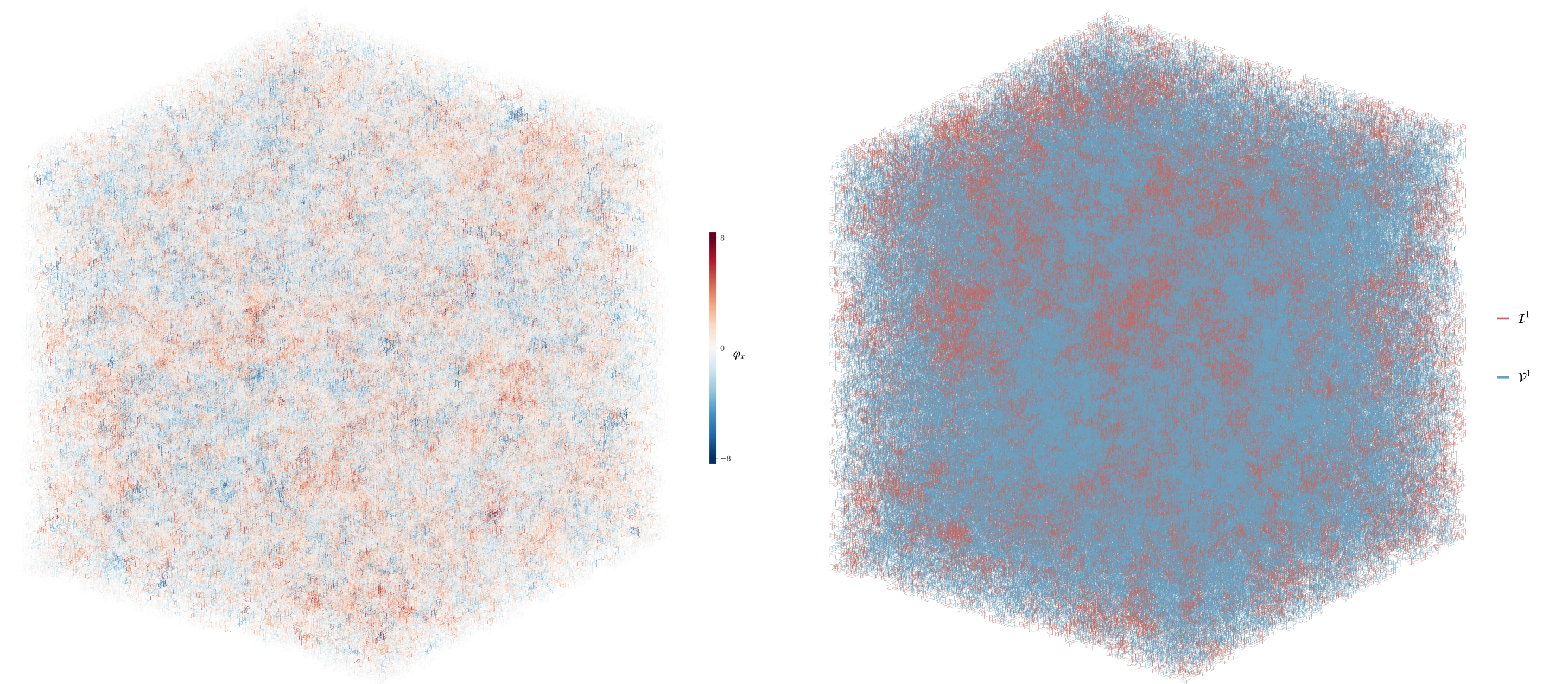


Fig. 1: Gaussian free field (left) and random interlacements (right, with $u = 1$) on a supercritical Bernoulli cluster (simulation in a finite box in $d = 3$, with $p = 0.3 > p_c(3)$).

Note that a typical $\mathcal{C}_\infty$ does not fit into the class of general weighted graphs considered in Drewitz, Prévost, and Rodriguez 2025.

Theorem 2 (Phase transition for the vacant set).

There exists a deterministic critical value $u_*(d, p) \in (0, \infty)$ such that $\mathbb{Q}_p$ -almost surely, for each $u < u_*(d, p)$, $\mathcal{V}_\omega^u$ contains an infinite cluster $\mathbb{P}_\omega^I$ -almost surely, while for each $u > u_*(d, p)$, $\mathcal{V}_\omega^u$ contains only finite clusters $\mathbb{P}_\omega^I$ -almost surely.

- **Cutset argument.** Disconnection in the vacant set may be caused by interlacement vertices or by missing open connections in the underlying Bernoulli cluster. Thus, a blocking structure need not be formed by interlacements alone.
- **A preliminary coarse-graining and a multiscale renormalization.** A preliminary coarse-graining step is performed similarly to Barlow 2004 (see also Antal and Pisztora 1996), which enables us to work on a renormalized plane $\mathbf{F} = k(\mathbb{Z}^2 \times \{0\})^{d-2}$ where we have planar duality. We then apply **a second multiscale renormalization** based on annulus crossings in $\mathbf{F}$. Our **uniform estimate over tree embeddings** ensures that any blocking circuit in $\mathbf{F}$ forces interlacements to hit the frames of a positive proportion of the leaf boxes (see also Fig. 3).

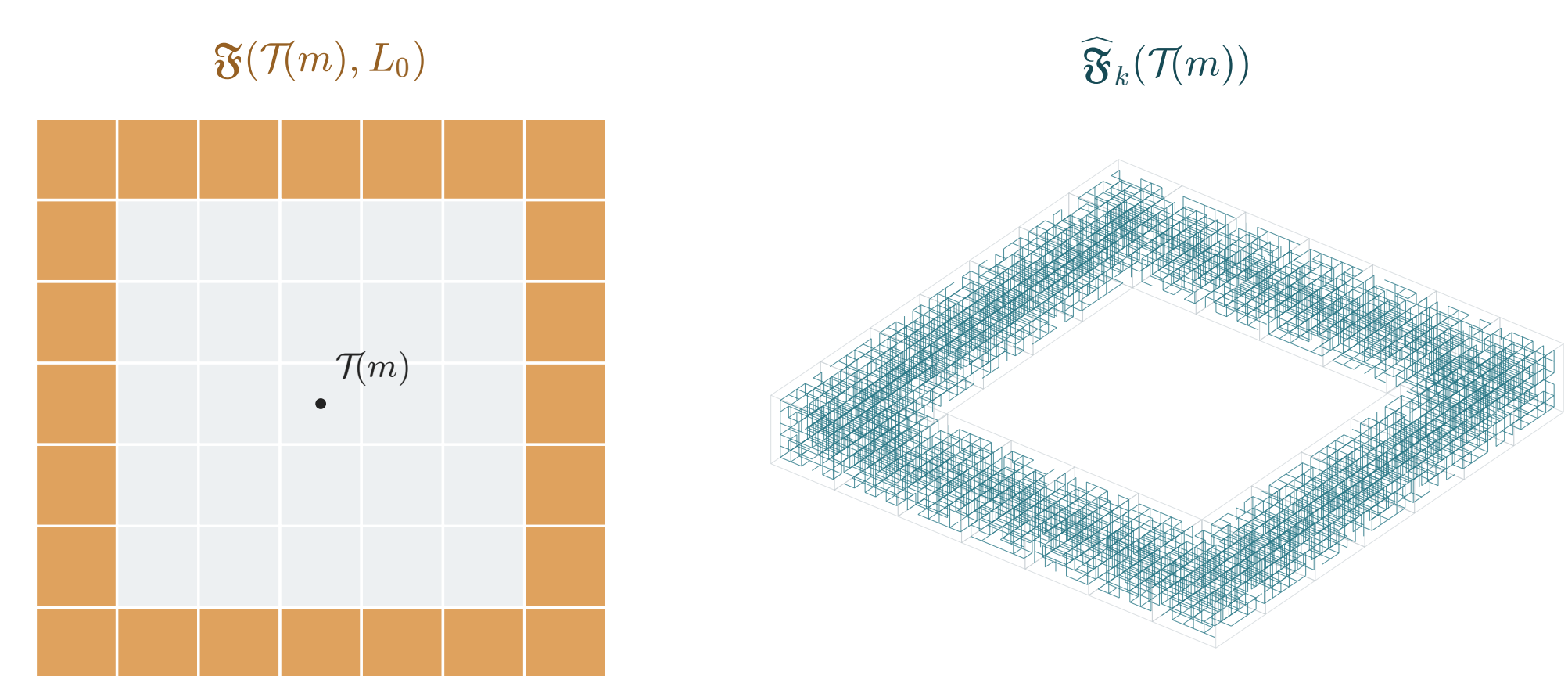


Fig. 3: Each box on the left corresponds to a vertex of $\mathbf{F}$. We are able to require a positive proportion of the frames of the leaf-boxes to be “open” in the renormalized site percolation.

- **Cost of interlacements hitting one-dimensional objects.** We use an argument similar to that in Ráth 2015, but with the frame in $\mathbb{Z}^2$ replaced by the thickened porous frame on the right of Fig. 3, to show that it is exponentially costly for interlacements to hit ρ^{2^n} thickened porous frames. The argument is based on a random walk construction of interlacements and the strong Markov property of the random walk.
- **Outline of the proof of $u_* > 0$.** For small u , long blocking circuits are unlikely, so vacant coarse-grained boxes percolate. Their connections lift to an infinite cluster in $\mathcal{V}_\omega^u$.

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Acknowledgements. This work is supported by Hong Kong RGC grants ECS 26301824 and GRF 16303825.